Journal of Applied Mathematics and Mechanics 73 (2009) 115–123

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

# Estimation of the time a perturbed Lagrangian system stays in an assigned region☆

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ARTICLE INFO

*Article history:*
Received 12 August 2008

ABSTRACT

The problem of estimating the mean time a weakly perturbed dynamical system stays in a fixed region of the phase variables is investigated. The motion is described by Lagrange's equations with an attractive force potential and in the presence of additive dissipative forces. The corresponding Cauchy problem is obtained in Hamiltonian variables for a non-linear first-order partial differential equation. Its classical positive solution specifies the action functional and the estimate sought for the time interval. The structure of the equations that allows of an explicit solution in terms of expressions for the kinetic and potential energy, as well as dissipative and dispersion matrices for a random Wiener-type perturbation, is established. The phenomenon of the escape of a phase point from different parts of the boundary of the region is investigated. Interesting problems of estimating the time for the inversion of the inner gimbal of a gyroscope, the time taken to reach an assigned level or a potential barrier of a multidimensional oscillatory system that has central symmetry, and the time a non-linear system with two degrees of freedom takes to escape over a potential barrier for a Henon–Heiles potential are investigated as examples.


The mean residence (confinement) time of a state vector in an assigned region of phase space is one of the fundamental characteristics that determines the reliability or "viability" of natural, physical, technical and other systems. The theory of large deviations[1,2] reduces the asymptotic calculation of the expected value of the time for weakly perturbed dynamical systems that are held by means of dissipative and attractive effects to solving the problem of minimizing a deterministic criterion that has the meaning of an action functional.

The construction of a functional as a function of perturbed Hamiltonian variables and the solution of the Cauchy problem for the corresponding Hamilton–Jacobi equation underlie the asymptotic analysis of the quality criterion required. The large deviations principle has been used in problems involving the control of escape of a phase point from an assigned region for weakly perturbed systems.[3,4] Using such an approach, we obtained estimates of the mean time a point stays in a rectangular[5] or circular[6] region on a geometric plane and we estimated the limiting possibilities of a linear regulator that controls the motion of the system.

## 1. Statement of the problem

We will examine a more general problem of the containment of a non-linear dynamical system in a phase region of arbitrary shape. The motion in dimensionless variables is described by Lagrange's equations of the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{\partial R}{\partial \dot{q}} = \varepsilon\sigma\dot{w}(t), \quad \sigma = \text{const}$$
$$L = T(q,\dot{q}) - \Pi(q), \quad T = (\dot{q}, M(q)\dot{q})/2, \quad R = (\dot{q}, B\dot{q})/2, \quad B = \text{const} \tag{1.1}$$

Here $q$ and $\dot{q}$ are vectors of the generalized coordinates and velocities, respectively, of dimension $n \geq 1$; $L$ is the Lagrange function; $R$ is the dissipative Rayleigh function; $w(t)$) is a standard Wiener process of dimension $l \geq 1$ and $\varepsilon$ is a small parameter. The symmetrical matrix $M$ of the kinetic energy $T$ is assumed to be non-degenerate, and the potential energy $\Pi$ is assumed to be a unimodal function that satisfies the conditions of a strict minimum

$$\Pi(0) = 0, \quad \partial\Pi(0)/\partial q = 0, \quad (\partial^2\Pi/\partial q^2\xi, \xi) > 0, \quad \xi \neq 0 \tag{1.2}$$

☆ *Prikl. Mat. Mekh.* Vol. 73, No. 2, pp. 163-175, 2009.
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0021-8928/$ – see front matter 
doi:10.1016/j.jappmathmech.2009.04.001

The motion of system (1.1), (1.2) is examined in the asymptotically long time interval $\tau^\varepsilon$ in the assigned, possibly unbounded region $G$ of the phase variables $q$ and $\dot q$ with boundary $\Gamma$ and closure $G^*$:

$$t_0 \le t < \tau^\varepsilon; \quad \tau^\varepsilon \to \infty, \quad \varepsilon \to 0$$
$$(q, \dot q) \in G = G_q \times G_{\dot q}, \quad G^* = G \cup \Gamma \tag{1.3}$$

To simplify investigations of the motion of the highly non-linear, stochastic system (1.1), we will confine ourselves to the case of perturbations of the generalized "white noise" type. The general case of Gaussian perturbations has been examined for systems in the Cauchy form,[2] and several special cases of the problem have been discussed.[5,6] The smoothness properties of the system and additional requirements are formulated below.

We will use the Lagrangian form of system (1.1) and convert it into a system of perturbed Hamilton equations using a standard transformation.[7] We change from the Lagrange variables $q$, $\dot q$, $t$ to the Hamilton variables $q$, $p$, $t$ by means of the replacement of variables[11]

$$q = q, \quad p = \partial L/\partial \dot q, \quad \dot q = M^{-1}(q)p, \quad t = t \tag{1.4}$$

according to which the sets $G$, $G^*$ and $\Gamma$ (1.3) are changed.

Using transformation (1.4), we write out the unperturbed Hamilton function[11]

$$H(q, p) = (\dot q, p) - L(q, \dot q) \equiv T^*(q, p) + \Pi(q)$$
$$T^* = T(q, M^{-1}(q)p) = (p, M^{-1}(q)p) \tag{1.5}$$

The unperturbed system (1.1) (with $B = 0$ and $\sigma = 0$) is conservative: $H = \text{const}$. In the general case $H = E$ can be regarded as the perturbed energy, and $\dot E < 0$ for $\varepsilon = 0$ and $\dot q \neq 0$ by virtue of the dissipativity of the Rayleigh function:[11] $\dot E = -2R$. According to formulae (1.4) and (1.5), system (1.1) is equivalent to the following

$$\dot q = \frac{\partial H}{\partial p}, \quad \dot p = -\frac{\partial H}{\partial q} - B\frac{\partial H}{\partial p} + \varepsilon\sigma\dot w$$
$$\dot E = -\left(\frac{\partial H}{\partial p}, B\frac{\partial H}{\partial p}\right) + \varepsilon\left(\frac{\partial H}{\partial p}, \sigma\dot w\right), \quad E = H(q, p) \tag{1.6}$$

Note that the equation for $q$ does not explicitly contain random perturbations, i.e., system (1.6) is degenerate according to the terminology of the theory of stochastic processes,[1,2] making it difficult to apply the large deviations principle directly. We will introduce assumptions that enable us to apply this principle in the case of a degenerate system.[2] It is assumed (see (1.3)) that the region of motion $G$ that is admissible for the variables $(q, p)$ is an open region in $R^{2n}$ with a smooth boundary $\Gamma$ and a closure $G^* = G \cup \Gamma$ and that the following conditions hold:

1) the function $H(q, p)$ is uniformly continuous and twice continuously differentiable in $G^*$;
2) the matrix $A = \sigma\sigma'$ is positive-definite (the prime denotes transposition);
3) the equalities

$$H = 0, \quad \partial H/\partial q = 0, \quad \partial H/\partial p = 0 \ (q = p = 0) \tag{1.7}$$

hold, the point $O$ being an isolated asymptotically stable equilibrium position of the unperturbed ($\varepsilon = 0$) system that lies strictly within the region $G$ outside a certain $\delta_\varepsilon$ neighbourhood of its boundary $\Gamma$ ($\delta_\varepsilon \to 0$ as $\varepsilon \to 0$);
4) the unperturbed system and the dissipative function are such that all trajectories that emerge from points in the region $G^*$ remain in this region, and the phase point approximates the equilibrium position (1.7) as $t \to +\infty$ without going outside the limits of this region.

Conditions 3 and 4 are constructive, since they were formulated for an unperturbed deterministic system. It follows from them that when there are no perturbations ($\varepsilon = 0$), the phase point $(q, p)$ remains within the region $G$ during the unbounded time $\tau^0 = \infty$. The presence of weak perturbations ($\sim\varepsilon$) can result in rare escapes of the system from $G^*$, which is interesting and important from the theoretical and applied points of view.

We pose the problem of the asymptotic estimation for the mean value $\mathrm{M}\tau^e$ of the time $\tau^e$ that the phase point $(q, p)$ stays within the region $G$

$$\tau^\varepsilon = \inf\{t : (q(t), p(t)) \in G\}, \quad (q(\tau^\varepsilon), p(\tau^\varepsilon)) \in \Gamma \tag{1.8}$$

According to relation (1.8), $\tau^e$ is the time when the boundary $\Gamma$ is first reached, and M is the mathematical expectation operator for all forms $\{\dot w(t)\}$ of the perturbation $w(t)$.

The mathematical theory has been developed,[2] and it has been proved that the following asymptotic estimate holds when conditions 1-4 are satisfied:

$$\lim_{\varepsilon \to 0}[\varepsilon^2 \ln(\mathrm{M}\tau^\varepsilon)] = \min_{(q, p) \in \Gamma} \Phi(q, p) = \Phi_0 \tag{1.9}$$

The quantity

$$\mathrm{M}\tau^\varepsilon \sim \exp(\varepsilon^{-2}\Phi_0), \quad \varepsilon \ll 1 \tag{1.10}$$

can be taken as a required estimate.

According to estimate (1.9), the function $\Phi(q, p)$ is the minimum of the action functional[2] that can be reached on the escape trajectory $(q(t), p(t))$ from the region $G$. It is specified by the solution of the Hamilton–Jacobi equation for the deterministic variational problem. We recall that all the quantities are represented in dimensionless form.

The corresponding non-linear Cauchy problem for the unknown function $\Phi$ is described by a non-linear, partial differential equation and by the initial condition[1,2]

$$\left(\frac{\partial H}{\partial p}, \frac{\partial \Phi}{\partial q}\right) - \left(\frac{\partial H}{\partial q}, \frac{\partial \Phi}{\partial p}\right) - \left(B\frac{\partial H}{\partial p}, \frac{\partial \Phi}{\partial q}\right) + \frac{1}{2}\left(\frac{\partial \Phi}{\partial p}, A\frac{\partial \Phi}{\partial p}\right) = 0$$
$$(q, p) \in G; \quad \Phi(0, 0) = 0 \tag{1.11}$$

It is necessary to find the positive-definite function $\Phi$

$$\Phi(q, p) > 0, \quad (q, p) \neq 0 \tag{1.12}$$

which presents considerable analytical and computational difficulties.

Note that the first two terms in Eq. (1.11) are Poisson brackets of the functions $H$ and $\Phi$. The solution of problem (1.11), (1.12) specifies only a necessary condition for a minimum of the action functional $\Phi$. If a classical (continuously differentiable) solution of the Cauchy problem[1–4,8] exists in the region $G^*$, it would specify a unique solution of the variational problem sought and asymptotic estimates (1.9) and (1.10). Non-uniqueness can occur for a non-smooth system, then the solution found will be an upper estimate of the action functional.

In the general case of system (1.1) or (1.6), exact or approximate solutions of the Cauchy problem (1.11), (1.12) lead to considerable computational difficulties. They can be overcome under several simplifying assumptions regarding the dimension and structure of equalities (1.6) and the shape of the region. The estimates (1.9) and (1.10) of the residence time $\tau^e$ were calculated for linear second-order systems when the region $G$ has a rectangular[5] or circular[6] shape. An analysis of perturbed Lagrangian systems[7,8] is of specific interest for problems related to the dynamics and control of mechanical systems under the conditions of uncertainty or counteraction.

We will give a meaningful interpretation of the Cauchy problem (1.11), (1.12). It means that the properties of the weakly perturbed system (1.6) enable us to confine ourselves to an investigation of the trajectories emerging from the point $O$ of the asymptotically stable equilibrium position of the unperturbed system. As we know,[1] when $\varepsilon > 0$ is sufficiently small and conditions 3 and 4 hold, the phase point on a trajectory of the unperturbed system (1.6) that starts at an arbitrary point $(q^0, p^0) \in G$ is trapped in a small $\rho_\varepsilon$ neighbourhood of the point $O$ ($\rho_\varepsilon \to 0$ as $\varepsilon \to 0$) and moves under the action of stabilizing and perturbing forces until it arrives at the boundary $\Gamma$ of the region. The time intervals of attraction to the point $O$ and escape from its vicinity to the boundary $\Gamma$ are estimated to be quantities of the order of unity. They are negligibly small compared with the length of the time interval for motion near the point $O$, which is of the order of exp $\varepsilon^{-2}$. This time interval is regarded as an estimate of the time the phase point stays within the region of attraction $G$. This estimate is valid for all initial conditions within $G$ outside the $\delta_\varepsilon$ neighbourhood of its boundary $\Gamma$.

## 2. Investigation of the Cauchy problem

We will construct the non-trivial solution sought using the structural properties of system (1.6). Direct substitution proves that the Cauchy problem (1.11), (1.12) is formally solvable in the form of a continuously differentiable function $\Phi(q, p)$, which satisfies that the following system of $2n$ differential equalities:

$$\frac{\partial \Phi}{\partial q} = K\frac{\partial H}{\partial q}, \quad \frac{\partial \Phi}{\partial p} = K'\frac{\partial H}{\partial p}, \quad K' = 2A^{-1}B, \quad (q, p) \in G \tag{2.1}$$

In fact, the first two terms (Poisson brackets) in (1.11) are mutually cancelled for an arbitrary $n \times n$ matrix $K$. The next two terms are also cancelled if expression (2.1), where $A$ and $B$ are symmetrical positive-definite matrices, is taken as the matrix $K$. The matrix $K$ characterizes the effect of dissipation and random perturbations on the functional $\Phi$. Note that in the general case the functional $\Phi$ is not symmetrical.

The construction of the smooth function $\Phi(q, p)$, based on $2n$ differential equalities of type (2.1), requires the integrability (consistency) conditions.[12] They can be reduced to the three systems of identities

$$\frac{\partial^2 \Phi}{\partial q_j \partial q_i} = \frac{\partial^2 \Phi}{\partial q_i \partial q_j}, \quad \frac{\partial^2 \Phi}{\partial p_j \partial p_i} = \frac{\partial^2 \Phi}{\partial p_i \partial p_j}, \quad \frac{\partial^2 \Phi}{\partial q_j \partial p_i} = \frac{\partial^2 \Phi}{\partial p_i \partial q_j} \tag{2.2}$$

for all values of the indices $i, j = 1, \ldots, n$. Of course, equalities (2.2) impose structural constraints on the function $H(q, p)$ and the matrix $K$. In particular, according to expression (2.1), the second system of equalities in (2.2) leads to the representations

$$\frac{\partial \Phi}{\partial p} = K'\frac{\partial T^*}{\partial p} = K'M^{-1}p, \quad K'M^{-1}(q) \equiv M^{-1}(q)K$$
$$\Phi(q, p) = \frac{1}{2}(p, K'M^{-1}(q)p) + \Theta(q) \tag{2.3}$$

Relations (2.3) contain a matrix identity for $M$ and $K$ and a scalar function $\Theta$, to be determined from the first equality in (2.1). This leads to additional structural constraints. There is a system of $n$ linear partial differential equations with the initial condition

$$\frac{\partial \Theta}{\partial q} = K\frac{\partial \Pi}{\partial q}, \quad \Theta(0) = 0; \quad \Theta(q) > 0, \quad q \neq 0 \tag{2.4}$$

The first system of equalities in (2.2) and expression (2.4) for the first derivatives give $n(n-1)/2$ identities and the representation for $\Theta$

$$\frac{\partial^2\Theta}{\partial q_j\partial q_i} = \sum_{r=1}^{n} k_{ir}\frac{\partial^2\Pi}{\partial q_j\partial q_r} \equiv \frac{\partial^2\Theta}{\partial q_i\partial q_j} = \sum_{r=1}^{n} k_{jr}\frac{\partial^2\Pi}{\partial q_i\partial q_r}$$
$$\Theta(q) = L[\Pi] \tag{2.5}$$

Here $L$ is a linear operator (see below).

We will present several special cases of system (1.6) that are encountered in applied research, for which the integrability (consistency) requirements hold.

1°. The conditions written down above automatically hold for scalar non-linear system (1.6). From (2.3)–(2.5), we obtain

$$\Phi(q,p) = \kappa H(q,p), \quad \kappa = 2a/b, \quad A = (a), \quad B = (b) \tag{2.6}$$

The value of $\Phi_0$ is determined by applying minimization procedure (1.10) to the function $\Phi$ (2.6) at a uniform set of boundary points (see below).

2°. Let $K$ (2.1) be a scalar matrix for a certain system (1.6). Then we have

$$K = \kappa I_n, \quad \Theta(q) = \kappa\Pi(q)$$
$$\Phi(q,p) = \kappa(p, M^{-1}(q)p)/2 + \kappa\Pi(q) = \kappa H(q,p) \tag{2.7}$$

Further derivations require minimization of the Hamilton function $H$ with respect to $(q,p)\in\Gamma$ according to estimate (1.10) (see below).

3°. Consider the case of the diagonal matrix $K=\mathrm{diag}(\kappa_1,\ldots,\kappa_n)$. In this case integrability condition (2.5) unquestionably holds if $\partial^2\Pi/\partial q_i\partial q_j \equiv 0$ for $i \neq j$. Also taking into account the symmetry condition (2.3) for the matrix $KM^{-1}$, we obtain the representation

$$\Phi(q,p) = \frac{1}{2}(p, KM^{-1}(q)p) + \sum_{i=1}^{n}\kappa_i f_i(q_i)$$
$$\Pi(q) \equiv \sum_{i=1}^{n} f_i(q_i), \quad f_i(0) = f_i'(0) = 0, \quad i = 1, 2, \ldots, n \tag{2.8}$$

Note that this symmetry condition holds if the matrix $M(q)$ is diagonal; then $K'M \equiv M^{-1}K$. As a result, the first term in the expression for $\Phi$ can be written in the form of a sum

$$\frac{1}{2}(p, KM^{-1}(q)p) = \frac{1}{2}\sum_{i=1}^{n}\frac{\kappa_i}{m_i(q)}p_i^2, \quad M = \mathrm{diag}(m_1,\ldots,m_n) \tag{2.9}$$

Substitution of (2.9) into the first equality in (2.8) leads to the required expression for the action functional $\Phi(q, p)$, which is then minimized according to estimate (1.9), (1.10) (see below).

4°. Consider the intermediate situation for the case of a diagonal matrix $K$ of the form

$$K = \mathrm{diag}(\kappa_1,\ldots,\kappa_m,\ldots,\kappa_n); \quad \kappa_i \neq \kappa_j \ (i \neq j,\ i, j = 1,\ldots, m \le n-2)$$
$$\kappa_{m+1} = \kappa_{m+2} = \ldots = \kappa_n, \quad n \ge 3 \tag{2.10}$$

A "combination" of conditions 2° and 3° occurs (see below). When expressions (2.7) and (2.8) are taken into account, consistency conditions (2.4) and (2.5) lead to the relations

$$\Pi(q) = \sum_{i=1}^{m} f_i(q_i) + \Psi(q_{m+1},\ldots,q_n), \quad KM^{-1} = M^{-1}K$$
$$\Phi(q,p) = \frac{1}{2}(p, KM^{-1}(q)p) + \sum_{i=1}^{m}\kappa_i f_i(q_i) + \kappa_n\Psi(q_{m+1},\ldots,q_n) \tag{2.11}$$

The ensuing investigations of $\Phi$ (2.11) are analogous to the investigations described above.

5°. Consider the special case of a linear system of the form (1.6), where $M=\mathrm{const}$ and $\Pi=(q, Cq)/2$. For the function $\Phi$, similar techniques give the representation

$$\Phi(q,p) = (p, KM^{-1}p)/2 + (q, KCq)/2 \tag{2.12}$$

under the condition that the matrices $KM^{-1}$ and $KC$ are symmetrical.

When there are no structural properties, the solution of the Cauchy problem (1.11), (1.12) presents considerable difficulties. They are due to the non-linearity of the equation and the fact that $\Phi \equiv 0$ is a solution. There are no regular analytical and numerical methods for investigating Cauchy problems.

## 3. Calculation of the limit of the action functional

We will determine the value of $\Phi_0$ (1.9), (1.10), i.e., the minimum of the smooth function $\Phi(q, p)$ on the boundary $\Gamma$ for various ways of assigning the boundary points.

The limit $\Phi_0$ is calculated directly if the values $q^*$ and $p^*$ of the variables $(q, p)$ are fixed on the boundary $\Gamma$. Then the quantity sought is $\Phi_0 = \Phi(q^*, p^*)$, where $(q^*, p^*) \neq 0$. The physical picture includes a situation in which $p^* = 0$, and the value of $q^*$ corresponds to the maximum of the function $\Pi(q)$, i.e., "overcoming the potential barrier $\Pi(q^*)$."

As a rule, the boundary is described by the system of equalities

$$\Gamma = \{q, p : F(q, p) = 0\}, \quad F = (F_1, \ldots, F_l), \quad l \le 2n \tag{3.1}$$

The vector function $F$ is assumed to be twice continuously differentiable in the convex, simply connected region $G$. The function $\Phi(q, p)$ under condition (3.1) is minimized using a standard procedure based on Lagrange multipliers.[13] For it we write out the expanded function

$$\Lambda(q, p, \lambda) = \Phi(q, p) + (\lambda, F(q, p)) \tag{3.2}$$

for which the unconditional minimum with respect to $q$, $p$ (and $\lambda$) is determined, where $\lambda$ is the vector of the Lagrange multipliers. The necessary conditions for an absolute minimum of the function $\Lambda$ (3.2) lead to the system of $2n + l$ non-linear equations in the unknowns $q$, $p$ and $\lambda$

$$\partial\Lambda/\partial q = 0, \quad \partial\Lambda/\partial p = 0, \quad \partial\Lambda/\partial\lambda = F(q, p) = 0 \tag{3.3}$$

It is assumed that the values $q^*$, $p^*$ and $\lambda^*$ of the unknowns are determined using to equalities (3.3). Verifying the sufficient conditions for a local minimum reduces to determining that the matrix of the second derivatives of $\Lambda$ with respect to the vector $r = (q, p)$ is positive-definite under the additional condition that the quadratic form is positive on a linear manifold[11]

$$\left(\xi, \frac{\partial^2\Lambda}{\partial r^2}\xi\right) > 0, \quad \frac{\partial F}{\partial r}\xi = 0, \quad r = r^*, \quad \lambda = \lambda^* \tag{3.4}$$

The minimizing vector $r^* = (q^*, p^*)$ found from relations (3.3) and (3.4) specifies the phase point where system (1.6) reaches the boundary $\Gamma$: $\Phi_0 = \Phi(q^*, p^*)$. In the general case a set of escape points is allowed.

If the admissible region $G$ is determined by the vector of the generalized coordinates $q$, i.e., $F = F(q)$, and the momentum $p$ is not taken into account, then, according to what was said in Section 2, the procedure just described leads to the relations

$$\frac{\partial\Theta}{\partial q} + \left(\lambda\frac{\partial F}{\partial q}\right) = 0, \quad F(q) = 0, \quad l \le n, \quad p^* = 0$$
$$\Phi_0 = \Phi(q^*, 0) = \Theta(q^*) \tag{3.5}$$

An interesting property of the perturbed system follows from these relations: when no condition is imposed on the momentum $p$, crossing of the boundary, in particular, overcoming of the potential barrier, occurs with zero velocity.

The case of an ellipsoidal region $G$ with centre at the point $r = 0$ is interesting from the theoretical and applied points of view. This case is defined as follows:

$$G = \{q, p: (r, D^{-1}r) \le 1\}, \quad \Lambda = \Phi(q, p) + \lambda((r, D^{-1}r) - 1) \to \min_{r, \lambda} \tag{3.6}$$

Here $D$ is a symmetrical positive-definite matrix. The necessary conditions (3.3) for determining the vector $r = (q, p)$ and the multiplier $\lambda$ lead to the expressions

$$\frac{\partial\Phi}{\partial r} - \left(\frac{\partial\Phi}{\partial r}, r\right)D^{-1}r = 0, \quad (r, D^{-1}r) = 1, \quad \lambda = -\left(\frac{\partial\Phi}{\partial r}, r\right) \tag{3.7}$$

The vector $r^*$ is determined from the first $2n + 1$ relations. Vector equation (3.7) is simply degenerate, since scalar multiplication by $r \neq 0$ leads to an identity. In particular, if $\Phi$ is a quadratic form in $r$ of the form (2.12), then $\lambda = \lambda_j^*$, where the $\lambda_j^*$ are the real roots of the characteristic equation

$$\left| \left\| \begin{matrix} KM^{-1} & 0 \\ 0 & KC \end{matrix} \right\| + \lambda D^{-1} \right| = 0 \tag{3.8}$$

i.e., a polynomial of order $2n$ for system (3.7). Because the matrices $KM^{-1}$, $KC$ and $D^{-1}$ are symmetrical and positive-definite, Eq. (3.8) has $2n$ negative roots $\lambda_j^*$. From the corresponding set of eigenvectors $\lambda_j^*$, we select the eigenvector which gives the minimum value of $\Phi$:

$$\Phi_0 = \Phi(q_j^*, p_j^*) \to \min_j, \quad j = 1, \ldots 2n \tag{3.9}$$

Thus, the solution of problem (3.6) is equivalent to algebraic problem (3.8), (3.9). Note that in the general case the matrix $D$ is not positive-definite and that the region $G$ is an elliptical cylinder. As above (see relation (3.5)), suppose the boundary $\Gamma$ does not depend on the momentum $p$. Then we have

$$\Gamma = \{q, p : (q, D_q^{-1} q) = 1\}, \quad p^* = 0; \quad \left|KC + \lambda D_q^{-1}\right| = 0$$
$$\Phi_0 = \Phi(q_j^*, 0) \to \min_j, \quad j = 1, \dots, n \tag{3.10}$$

Note that in case of the corresponding matrices are similar, any point $(q, p) \in \Gamma$ is an escape point.

Effective solutions of examples that are meaningful in mechanics are presented below (see Sections 4–6). They illustrate the constructiveness of the examples presented in Section 2.

## 4. Flipping of the inner gimbal of a free gyroscope

We will use the results obtained in Sections 2 and 3 to estimate the mean rotation time of the rotor of an immobile balanced gyroscope in an assigned regime. The case of a harmonic perturbation was previously investigated.[14] We will represent the equation of rotation of the inner gimbal (ring) under the action of dissipative and stochastic torques in the form (1.1)

$$\ddot{\beta} + \nu^2 dV/d\beta = -b\dot{\beta} + \varepsilon\sigma\dot{w}(t); \quad V(\beta) = \frac{(l - \sin\beta)^2}{2(1 - c\sin^2\beta)}, \quad |l| < \infty, \quad 0 < c < 1 \tag{4.1}$$

Here $\beta$ is the angle of rotation angle of the inner gimbal, and $\nu$ is the angular frequency of the nutational oscillations, which is generally specified by the high rate of angular rotation of the rotor. The function $V(\beta) \geq 0$ has the meaning of the reduced "potential energy," which is $2\pi$-periodic in $\beta$. The parameter $l$ is the ratio of the generalized momenta, i.e., the angular momenta about the axes of the outer gimbal and the gyroscope rotor, respectively, which are the integrals of motion when there are no perturbations with respect to these axes.[14] To be specific, we can set $l > 0$ without loss of generality of the problem. The parameter $c$ is specified by the ratio of the reduced moments of inertia and satisfies constraints (4.1). An additional control torque that contains positive feedback for the orientation angle $\beta$ and the angular velocity $\dot{\beta}$ can be applied about the axis of rotation of the inner gimbal.

We will first investigate the motion of system (4.1) in a vicinity of the stable equilibrium position $\beta^*$ of the unperturbed system (when $\varepsilon = 0$). For values of $l$ that are of practical importance ($|l| < 1$), we have $V(\beta) = 0$ when $\sin\beta = l$, i.e., there is a zero minimum; to be specific, we will take the value $\beta^* = \arcsin l$. In the interval $-\pi < \beta \leq \pi$ (or $0 \leq \beta < 2\pi$) the function $V(\beta)$ has a zero value when $\beta^* = \pi - \beta^*$, as well as the two local maximum values $V(\pm\pi/2)$.

Thus, perturbed motion in the vicinity of $\beta = \beta^*$ is considered. The average time for escaping the potential barrier, i.e., for the inner gimbal to flip, is specified by the value of the function

$$V(\pi/2) = (l - 1)^2(1 - c)^{-1}/2 < V(-\pi/2) = (l + 1)^2(1 - c)^{-1}/2 \tag{4.2}$$

and is calculated using formulae (1.10) and (3.5) and expression (2.6) (or (2.7) for $n = 1$). As a result, when relations (4.2) are taken into account, we obtain the following expression for the coefficient $\Phi_0$ in the exponent in (1.10)

$$\Phi_0 = \Phi(\pi/2, 0) = 2b\sigma^{-2}\nu^2 V(\pi/2) = b\sigma^{-2}\nu^2(l - 1)^2(1 - c)^{-1} \tag{4.3}$$

It follows from an analysis of $\Phi_0$ that $l = 1$ is the critical value of $l$. If $l > 1$, the following values are subject to analysis[12]

$$\rho^* = \arcsin(cl)^{-1}, \quad cl > 1, \quad V(\beta^*) = (l^2 - 1/c)/2 > 0$$
$$V(\beta^*) < V(\pm\pi/2) = (l \mp 1)^2(1 - c)^{-1}/2 \tag{4.4}$$

The difference $V(\pi/2) - V(\beta^*)$, in which these values are calculated using to relation (4.4), determines the minimum $\Phi_0$ of the action functional $\Phi$ when the potential barrier is crossed. Note that, as was indicated in Section 3, crossing of the potential barrier occurs with zero velocity.

## 5. The Retantion of a multidimensional non-linear oscillator in the vicinity of a stable equilibrium

Consider the model example of the motion of a non-linear oscillator (point) in the vicinity of its zero position. We will assume that the potential energy $\Pi$ is specified by an expression of the special form

$$\Pi(R) = U(r) \equiv \frac{c}{2}r^2 + \frac{\lambda}{3}r^3 + \frac{\gamma}{4}r^4, \quad 0 \leq r = |R| < \infty$$
$$c > 0, \quad \lambda, \gamma \gtrless 0, \quad R = (x_1, \dots, x_n) \in R^n, \quad n \geq 1 \tag{5.1}$$

The function $U$ corresponds to the potential energy of a weightless spring with non-linear physical elasticity characteristics.

Without loss of generality, the mass $m$ of the point and the elasticity coefficient $c$ (when $c > 0$) can be set equal to unity. We will consider the case of an anisotropic viscous medium with a non-diagonal matrix $B$ and a multidimensional random process $w(t)$:

$$\ddot{R} + \frac{dU}{dr}\frac{R}{r} = -B\dot{R} + \varepsilon\sigma\dot{w}, \quad B' = B, \quad A = \sigma\sigma' = A' \tag{5.2}$$

It follows from relations (5.1) and (5.2) that the elastic force $-\partial\Pi/\partial R$ is in the opposite direction to the radius vector $R$ and that the additivity property ($n \geq 2$) of the type (2.6) obviously breaks down when $\lambda \neq 0$ and/or $\gamma \neq 0$. According to relations (1.4), the generalized momentum $P = \dot{R}$, and, according to equalities (2.1)–(2.3), the function $\Phi$ is given by the expressions

$$\Phi(R, P) = (P, KP)/2 + \Theta(R), \quad K = 2A^{-1}B = K'$$
$$\frac{\partial\Theta}{\partial R} = K\frac{\partial U}{\partial R} = \frac{1}{r}\frac{dU}{dr}KR, \quad \Theta(0) = 0; \quad K = (\kappa_{ij}) \tag{5.3}$$

Here and below, satisfaction of the integrability condition $K = K'$ is required.

Checking of the consistency (solvability) conditions of the type (2.4) of the vector equation for $\Theta$ (5.3) leads, according to relations (2.4), to the identities

$$\frac{d}{dr}\left(\frac{1}{r}\frac{dU}{dr}\right)\sum_{l=1}^{n}\kappa_{il}R_lR_j \equiv \frac{d}{dr}\left(\frac{1}{r}\frac{dU}{dr}\right)\sum_{l=1}^{n}\kappa_{jl}R_lR_i, \quad i \neq j \tag{5.4}$$

Relations (5.4) hold if $\kappa_{ij} = 0$ for $i \neq j$ and $\kappa_{ii} = \kappa_{jj}$ for $i, j = 1, 2, \ldots, n$, which correspond to the case in Section 1° (see identity (2.5)). As a result, the expressions for $\Theta$ and $\Phi$ have the form

$$\Theta(R) = \kappa U(r), \quad \Phi(R, P) = (\kappa/2)(P^2 + U(R)) \tag{5.5}$$

Identity (5.4) also holds if the multiplier in front of the summation sign in (5.4) is identically equal to zero. This requirement leads to the equality $\lambda = \gamma = 0$, i.e., $U = cr^2/2$. In this very simple case the potential energy is an additive function of the components of the vector $R$ (see formula (2.7) in Section 2, 3°). The diagonal structure of the matrix $K = \mathrm{diag}(\kappa_i)$ is allowed, and the function sought is $\Theta(R) = c(R, KR)/2$. Note, however, that system (5.2) will have the linear form (2.12) when $\lambda = \gamma = 0$ (see Section 2, 5°). Therefore, the expression obtained holds for an arbitrary positive-definite symmetrical matrix $K = (\kappa_{ij})$, i.e., we have representation (2.12) for $\Phi$:

$$\Phi(R, P) = [(P, KP) + c(R, KR)]/2, \quad K = 2A^{-1}B = K' \tag{5.6}$$

Then, according to relations (1.5) and the results in Section 3, the minimum $\Phi_0$ of the expressions for $\Phi(R, P)$ (see formulae (5.5) and (5.6)) on the specified boundary $\Gamma$ must be found. To be specific, we will consider the problem of reaching the fixed value $U^*$ of the potential energy: $U(R) = U^*$ by the phase point $(R, P)$. According to what was said in Section 3 (see formula (3.5)), since the magnitude of the momentum $P$ (the velocity $\dot{R} = P$) is not assigned, the minimizing value is $P = 0$.

We will investigate the behaviour of the potential energy $U(r)$ (5.1) for various values of the parameters $c$, $\lambda$ and $\gamma$. In the standard situation the first term is decisive for relatively small values of $r$. If $\lambda, \gamma \geq 0$, the function $U(r)$ will also be a monotonically increasing function and will increase faster than a parabola for relatively large $r$. The qualitative nature of the curve is maintained if one or two parameters are set equal to zero. In these cases, the value $r^*$ of the argument (the root) in the expression $U(r) = U^*$ is determined and is found by standard methods.

More diverse behaviour of $U(r)$ is observed when the parameters $c > 0$ and $\lambda$ and/or $\gamma$ have different signs. We will determine the extrema and points of inflection; the following relations hold

$$dU/dr = r(c + \lambda r + \gamma r^2) = 0, \quad d^2U/dr^2 = c + 2\lambda r + 3\gamma r^2 = 0 \tag{5.7}$$

It follows from the first relation in (5.7) that the value $r_0 = 0$ corresponds to the point of local minimum. The other extremum values of the argument $r$ are specified by the expressions

$$r^*_{1,2} = (2\gamma)^{-1}(-\lambda \pm (\lambda^2 - 4c\gamma)^{1/2}) > 0, \quad \gamma \neq 0, \quad 4c\gamma \leq \lambda^2$$
$$r^* = -c/\lambda, \quad \gamma = 0, \quad \lambda < 0; \quad r^* = (-c/\gamma)^{1/2}, \quad \lambda = 0, \quad \gamma < 0 \tag{5.8}$$

from which it follows that for $\gamma < 0$ there is one local maximum point, regardless of the magnitude and sign of $\lambda$. More specifically, $r^* = r^*_2 > 0$ when $\lambda \geq 0$, since $r^*_1 < 0$; in addition, $r^* = r^*_2 > 0$ when $\lambda \leq 0$; in both cases $\gamma < 0$. Similar behaviour of $U(r)$ is observed in the case when $\gamma = 0$, $\lambda < 0$ (see expression (5.8)).

The height of the potential barrier equals $U^* = U(r^*)$. For $\lambda = 0$, $\gamma < 0$ or $\lambda < 0$, $\gamma = 0$ it determines the minimum of the action functional $\Phi$ (5.5), which characterizes the mean time the oscillator stays in the $r^*$ neighbourhood of the point of attraction $r_0 = 0$. When $r > r^*$ the function $U(r)$ decreases monotonically without limit: $U(r) \to -\infty$.

In the case of one maximum point $r^*$, there is one point of inflection $r^{**} < r^*$. It is specified by Eq. (5.8) and corresponds to $r^{**}_2$:

$$r^{**}_{1,2} = (3\gamma)^{-1}(-\lambda \pm (\lambda^2 - 3c\gamma)^{1/2}) > 0, \quad \gamma \neq 0, \quad 3c\gamma \leq \lambda^2$$
$$r^{**} = -c/(2\lambda), \quad \gamma = 0, \quad \lambda < 0; \quad r^{**} = (-c/(3\gamma))^{1/2}, \quad \lambda = 0, \quad \gamma < 0 \tag{5.9}$$

Consider a more complex situation that arises when $\lambda < 0$ and $\gamma > 0$. Then, for the values $0 < 4c\gamma < \lambda^2 < \infty$ there are two additional extremum points $r^*_{1,2}$ (see expressions (5.8)). The value $r^*_1(r^*_1 > r^*_2)$ corresponds to the second local minimum $U(r^*_1)$, and $r^*_2$ corresponds to the local maximum $U(r^*_2)$, as was established above. According to relations (5.9), there are two inflection points $r^{**}_{1,2}$, where $0 < r^{**}_2 < r^*_2 < r^{**}_1 < r^*_1 < \infty$.

If $3c\gamma < \lambda^2$, there are no inflection points. When $4c\gamma > \lambda^2$, the graph of $U(r)$ increases monotonically as $r$ increases. It is noteworthy that the second inflection point and the points of the local extrema are identical if $4c\gamma = \lambda^2$. When the equality $c\gamma = (2/9)\lambda^2$ holds, the second local minimum $U(r^*_1) = 0$; if $c\gamma l \lessgtr (2/9)\lambda^2$ and $\lambda < 0$, this minimum is less than or greater than zero, respectively. The height of

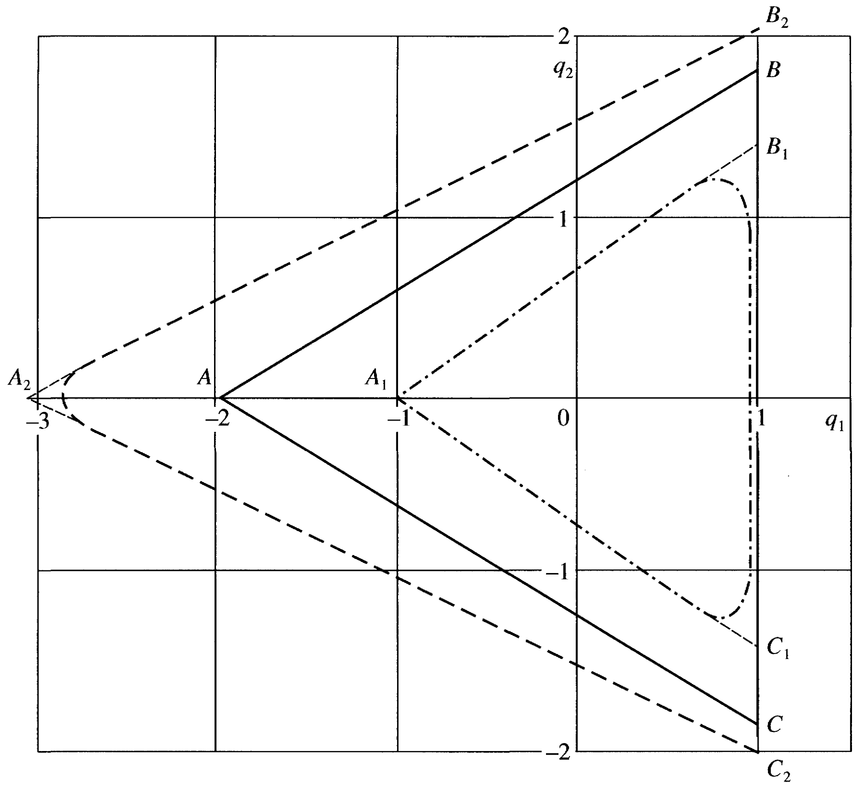


**Fig. 1.**

the potential barrier, which determines the time taken to transfer to a new state of motion, is specified by the values of the local minima $U(0) = 0$, $U(r_1^*) \lesseqgtr 0$ and of the local maximum $U(r_2^*) > 0$.

## 6. The retention of a system with a Henon–Heiles potential

We will evaluate the mean escape time of a specific system with two degrees of freedom from an admissible planar region. Consider the case in which this escape is associated with the crossing of a potential barrier by the trajectory of a particle (a point mass).[7–10] Examples include diverse physical processes: the synchronization of power plants, the confinement of elementary particles in an electromagnetic or optical trap, etc. The crossing of the potential barrier leads to instability or a transition into the region of attraction of another stable state and causes disruption of the normal function and/or a change in the physical properties of the system, as was pointed out in Sections 4 and 5. Therefore, the taken to reach the potential barrier may be regarded as one of the fundamental parameters of the reliability or "viability" of the system.

We will estimate this time for a system with a Henon–Heiles potential. It is used in an extensive list of problems concerning the motion of particles in potential traps of various kinds that have cylindrical symmetry.[9,10] The potential function $\Pi(q)$ is brought into the form

$$\Pi(q_1, q_2) = \frac{1}{2}\left(\rho^2 q_1^2 + q_2^2 - q_1 q_2^2 + \frac{1}{3}q_1^3\right) \tag{6.1}$$

The parameter ρ specifies the ratio of the frequencies of small oscillations a the linearized conservative system.

We will determine the positions of the special points and find the level surface corresponding to the potential barrier. The steady-state conditions have the form

$$\partial\Pi/\partial q_1 = \partial\Pi/\partial q_2 = 0$$

There are four stationary points: $O$ and $A$, $B$, $C$. The point $O$: $\{q_1 = q_2 = 0\}$ corresponds to the strict local potential minimum $\Pi_O = 0$, and the positions of the saddle points $A$, $B$ and $C$ and the corresponding potential levels are specified by the expressions

$$A : \{q_1 = -2\rho^2, q_2 = 0\}, \quad \Pi_A = \frac{2}{3}\rho^6$$

$$B, C : \{q_1 = 1, q_2 = \pm(1 + 2\rho)^{1/2}\}, \quad \Pi_{B,C} = \frac{1}{2}\left(\rho^2 + \frac{1}{3}\right) \tag{6.2}$$

It follows from conditions (6.2) that $\Pi_A < \Pi_{B,C}$ when $\rho > 1$ and $\Pi_A = \Pi_{B,C} = 2/3$ when $\rho = 1$. Thus, the level surface corresponding to the potential barrier touches the point $A$ when $\rho < 1$, the points $B$ and $C$ when $\rho > 1$, and the points $A$, $B$ and $C$ when $\rho = 1$. Fig. 1 shows the projections of the lines of intersection of the potential barrier with the potential function (6.1) onto the $(q_1, q_2)$ plane for various values of ρ: the boundaries of the admissible region for $\rho = 1$ (the solid line, $A = -2$, $B, C = \pm\sqrt{3}$), for $\rho = \sqrt{1/2}$ (the dash-dot line, $A = -1$, $B_1, C_1 = \pm\sqrt{2}$)

and for $\rho = \sqrt{3/2}$ (the dashed line, $A_2 = -3$, $B_2$, $C_2 = \pm 2$). The closed curves obtained with the singular points $A$, $B$ and $C$ (6.2) are interpreted as separatrices that separate the regions of stable and unstable motions.[7,9]

We will write the equations of perturbed motion in the form (1.1)

$$\ddot{q}_i + b\dot{q}_i + \partial\Pi/\partial q_i = \varepsilon\sigma\dot{w}_i(t), \quad i = 1, 2 \tag{6.3}$$

where the $w_i(t)$ are independent standard Wiener processes. We will determine the admissible region of motion for system (6.3). The total energy is

$$H(q, p) = (p_1^2 + p_2^2)/2 + \Pi(q_1, q_2), \quad p_i = \dot{q}_i, \quad i = 1, 2 \tag{6.4}$$

Let $\Pi^*$ be the value of the potential function that corresponds to $\Pi_A$ for $\rho < 1$, to $\Pi_{B,C}$ for $\rho > 1$ or to $\Pi_A = \Pi_{B,C} = 2/3$ for $\rho = 1$. If the total energy of system (6.4) $H(q, p) < \Pi^*$, the particle is held within the admissible region. In the $(q_1, q_2)$ plane the admissible regions are bounded by the separatrices (Fig. 1). Thus, the admissible region $G$ and its boundary $\Gamma$ are assigned in the form

$$G = \{q, p; H(q, p) < \Pi^*\}, \quad \Gamma = \{q, p; H(q, p) = \Pi^*\} \tag{6.5}$$

We will estimate the mean escape time from the region $G$ under the action of "weak" random perturbations of the normal process type. From relations (2.5) and (6.5) we obtain $\Phi(q,p) = \kappa H(q, p)$, where $\kappa = 2b/\sigma^2$ by virtue of Eqs. (2.1) and (6.3). Hence, from relations (1.9), (1.10) and (6.5), we obtain $\Phi_0 = (2b/\sigma^2)\Pi^*$. The approach described in Section 3 (see formulae (3.3)–(3.5)) shows that for $\rho < 1$ the point reaches the level of the potential barrier at the point $A$ (6.2). Then, for $\rho > 1$ the trajectory can cross the potential barrier at point $B$ or $C$ with equal probabilities. Finally, for $\rho = 1$ the trajectory can cross the potential barrier at point $A$, point $B$ or point $C$ with equal probabilities. In all cases the momenta $p_1 = p_2 = 0$ at the time when the potential barrier is reached.

## Acknowledgements

This research was financed by the Russian Foundation for Basic Research (08-01-00234, 08-01-00180 and 08-01-00068) and the Programme for the Support of Leading Scientific Schools (4315.2008.1).

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*Translated by* P.S.